POSITIVE RECURRENCE FOR A CLASS OF JUMP DIFFUSIONS

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Abstract: This work is concerned with asymptotic properties of a class of diffusion processes with jumps. In particular, we show that the property of positive recurrence is independent of the choice of the bounded domain in the state space. A sufficient condition for positive recurrence using Liapunov functions is derived.
Keywords: jump diffusion, Liapunov function, positive recurrence

1 INTRODUCTION

This work focuses on positive recurrence for a class of jump diffusion processes. Our motivation stems from the study of a family of Markov processes in which both continuous dynamics and jump discontinuity coexist. Such systems have drawn new as well as resurgent attention because of the urgent needs of systems modeling, analysis, and optimization in a wide variety of applications.

Asymptotic properties of diffusion processes and associated partial differential equations are well known in the literature. We refer to [2, 4] and references therein. Results for switching diffusion processes can be found in [7]. One of the important problems in stochastic systems is their longtime behavior. In the literature, criteria for certain types of weak stability including Harris recurrence and positive Harris recurrence for continuous time Markovian processes based on Foster-Liapunov inequalities were developed in [5]. Using results in that paper, some sufficient conditions for ergodicity of Lévy type operators in dimension one are presented in [6] under the assumption of Lebesgue-irreducibility. In a recent work [1], the authors discuss positive recurrence for jump processes with no diffusion part. Compared to the case of diffusion processes, even though the classical approaches such as Liapunov function methods

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and Dynkin's formula are still applicable for jump diffusion processes, the analysis is much more delicate because of the jump components. In contrast to the existing results, our new contributions in this paper are as follows. First, we study positive recurrence for a wide class of jump diffusions by using Khasminskii's approach developed in [4]. Second, we show that the property of positive recurrence is independent of the choice of the bounded domain in the state space. Moreover, we establish a sufficient condition for positive recurrence using Liapunov functions.

The rest of the paper is arranged as follows. Section 2 begins with the formulation of the problem. Section 3 presents our main results. Finally, Section 4 concludes the paper with further remarks.

2 FORMULATION

Notation: Throughout the paper, we use z' to denote the transpose of $z \in \mathbb{R}^{l_1 \times l_2}$ with $l_1, l_2 \geq 1$, and $\mathbb{R}^{d\times 1}$ is simply written as $\mathbb{R}^d$. If $x \in \mathbb{R}^d$, the norm of x is denoted by $|x|$. For any positive integer n, $B(0,n) := \{x \in \mathbb{R}^d \,|\, |x| < n\}$ is the open ball with radius n centered at the origin. The term *domain* in $\mathbb{R}^d$ refers to a nonempty connected open subset of the Euclidean space $\mathbb{R}^d$. If D is a domain in $\mathbb{R}^d$, then $\overline{D}$ is the closure of D, $D^c = \mathbb{R}^d - D$ is its complement. The space $C^2(D)$ refers to the class of functions whose partial derivatives up to order 2 exist and are continuous in D, and $C_b^2(D)$ is the subspace of $C^2(D)$ consisting of those functions whose partial derivatives up to order 2 are bounded. The indicator function of a set A is denoted by $\mathbf{1}_A$.

Let $b : \mathbb{R}^d \mapsto \mathbb{R}^d$, $\sigma : \mathbb{R}^d \mapsto \mathbb{R}^d \times \mathbb{R}^d$, and for each $x \in \mathbb{R}^d$, $\pi(x, dz)$ is a σ-finite measure on $\mathbb{R}^d$ satisfying

$$\int_{\mathbb{R}^d} (1 \wedge |z|^2)\pi(x, dz) < \infty.$$

For a function $f : \mathbb{R}^d \mapsto \mathbb{R}$ and $f \in C^2(\mathbb{R}^d)$, we define

$$\begin{aligned} \mathcal{L}f(x) &= \sum_{k,l=1}^{d} a_{kl}(x)\frac{\partial^2 f(x)}{\partial x_k \partial x_l} + \sum_{k=1}^{d} b_k(x)\frac{\partial f(x)}{\partial x_k} \\ &\quad + \int_{\mathbb{R}^d} \big(f(x+z) - f(x) - \nabla f(x)\cdot z\mathbf{1}_{\{|z|<1\}}\big)\,\pi(x, dz), \end{aligned} \tag{2.1}$$

where $a(x) = \sigma(x)\sigma'(x)$ and ∇f denotes the gradient of f.

Let $\Omega = D\big([0,\infty), \mathbb{R}^d\big)$ be the space of functions (mapping $[0,\infty)$ to $\mathbb{R}^d$) that are right continuous with left limits endowed with the Skorohod topology. Define $X(t) = w(t)$

for $w \in \Omega$ and let $\{\mathcal{F}_t\}$ be the right continuous filtration generated by the process $X(t)$. A probability measure $\mathbb{P}_x$ on Ω is called a solution to the martingale problem for $\big(\mathcal{L}, C_b^2(\mathbb{R}^d)\big)$ started at x if $\mathbb{P}_x(X(0)=x)=1$ and for every $f(\cdot) \in C_b^2(\mathbb{R}^d)$,

$$f(X(t)) - f(X(0)) - \int_0^t \mathcal{L}f(X(s))ds,$$

is a $\mathbb{P}_x$ martingale. If for each x, there is only one such $\mathbb{P}_x$, we say that the martingale problem for $\big(\mathcal{L}, C_b^2(\mathbb{R}^d)\big)$ is well-posed.

We assume the following conditions (A1)-(A4). Note that these conditions ensure that the martingale problem for $(\mathcal{L}, C_b^2(\mathbb{R}^d))$ is well-posed and the associated jump diffusion $X(t)$ is a strong Markov process (see [3]).

(A1) The functions $\sigma(\cdot)$ and $b(\cdot)$ are bounded and continuous.

(A2) There exists a constant $\kappa_1 \in (0,1]$ such that

$$\kappa_1|\xi|^2 \le \xi' a(x)\xi \le \kappa_1^{-1}|\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^d \text{ and } x \in \mathbb{R}^d.$$

(A3) For each $x \in \mathbb{R}^d$, $\pi(x,dz) = \widetilde{\pi}(x,z)dz$ is a σ-finite measure. Moreover, there is a constant $\kappa_2 > 0$ such that

$$\int_{\mathbb{R}^d} \big(1 \wedge |z|^2\big)\, \widetilde{\pi}(x,z)dz \le \kappa_2 \quad \text{for all } x \in \mathbb{R}^d.$$

(A4) For any $r \in (0,1)$, any $x_0 \in \mathbb{R}^d$, any $x, y \in B(x_0, r/2)$ and $z \in B(x_0, r)^c$, we have

$$\widetilde{\pi}(x, z-x) \le \alpha_r \widetilde{\pi}(y, z-y),$$

where α_r satisfies $1 < \alpha_r < \kappa_3 r^{-\beta}$ with κ_3 and β being positive constants.

Remark 2.1. Note that the measure $\pi(x,dz)$ can be thought of as the intensity of the number of jumps from x to $x+z$. Condition (A3) and (A4) tell us that $\pi(x,dz)$ is absolutely continuous with respect to the Lebesgue measure dz on $\mathbb{R}^d$, and the intensities of jumps from x and y to a point z are comparable if x, y are relatively far from z but relatively close to each other.

3 MAIN RESULTS

For simplicity, we introduce some notation as follows. For any $D \subset \mathbb{R}^d$, we define

$$\tau_D = \inf\{t \ge 0 : X(t) \notin D\}, \quad \sigma_D = \inf\{t \ge 0 : X(t) \in D\}.$$

Let $\beta_n = \inf\{t \geq 0 : |X(t)| \geq n\}$ be the first exit time of the process $X(t)$ from the bounded set $B(0,n)$. Then the sequence $\{\beta_n\}$ is monotonically increasing and $\beta_n \to \infty$ almost surely as $n \to \infty$. We will use this fact frequently in what follows. To proceed, we recall the definition of positive recurrence (see [4, 7]).

Definition 3.1. Suppose $D \subset \mathbb{R}^d$ is a bounded domain. A process $X(t)$ is said to be *positive recurrent* with respect to D if for any $x \in D^c$,

$$\mathbb{P}_x(\sigma_D < \infty) = 1 \quad \text{and} \quad \mathbb{E}_x[\sigma_D] < \infty.$$

We will establish certain preparatory results. The first one asserts that the process $X(t)$ will exit every bounded domain with a finite mean exit time.

Lemma 3.2. *Let $D \subset \mathbb{R}^d$ be a bounded domain. Then*

$$\sup_{x\in D} \mathbb{E}_x[\tau_D] < \infty. \tag{3.2}$$

Proof. By the uniform ellipticity condition in (A2), we have

$$\kappa_1 \leq a_{11}(x) \leq \kappa_1^{-1} \quad \text{for all } x \in D. \tag{3.3}$$

Let $f \in C_b^2(\mathbb{R}^d)$ be such that $f(x) = (x_1+\beta)^\gamma$ if $x \in \{y : d(y,D) < 1\}$, where the constants γ and β are to be specified and x_1 is the first component of x. Since D is bounded, we can choose constant β such that $1 \leq x_1 + \beta$ for all $x \in D$ and $f(x) \geq 0$ for all $x \in \mathbb{R}^d$. Let

$$\gamma := \frac{1}{\kappa_1}\left(\sup_{x\in D}\left[|b_1(x)|(x_1+\beta) + \kappa_2(x_1+\beta)^2\right] + 1\right) + 2 < \infty,$$

where κ_1 and κ_2 is the constants given in assumption (A2) and (A3). Then we have by (3.3) that

$$b_1(x)(x_1+\beta) + (\gamma-1)a_{11}(x) - \kappa_2(x_1+\beta)^2 \geq 1 \quad \text{for all} \quad x \in D. \tag{3.4}$$

Direct computation leads to

$$\begin{aligned}\mathcal{L}f(x) = {}& \gamma(x_1+\beta)^{\gamma-2}\left[b_1(x)(x_1+\beta) + (\gamma-1)a_{11}(x)\right] \\ &+ \int_{|z|\leq 1}\left[f(x+z) - f(x) - \nabla f(x)\cdot z\right]\widetilde{\pi}(x,z)(dz) \\ &+ \int_{|z|>1}\left[f(x+z) - f(x)\right]\widetilde{\pi}(x,z)(dz).\end{aligned} \tag{3.5}$$

Since $\gamma > 2$, f is convex on $\{x \in \mathbb{R}^d : d(x, D) < 1\}$. It follows that

$$\int_{|z|\le 1} \big[f(x+z) - f(x) - \nabla f(x) \cdot z\big]\widetilde{\pi}(x,z)(dz) \ge 0. \tag{3.6}$$

It is also clear that

$$\begin{aligned}\int_{|z|>1} \big[f(x+z) - f(x)\big]\widetilde{\pi}(x,z)(dz) &\ge -\int_{|z|>1} (x_1+\beta)^\gamma \widetilde{\pi}(x,z)(dz) \\ &\ge -\kappa_2 (x_1+\beta)^\gamma.\end{aligned} \tag{3.7}$$

It follows from (3.5), (3.6), (3.7), and (3.4) that $\mathcal{L}f(x) \ge \gamma$ for any $x \in D$. Let $\tau_D(t) = \min\{t, \tau_D\}$. Then we have from Dynkin's formula that

$$\begin{aligned}&\mathbb{E}_x f\left(X(\tau_D(t))\right) - f(x) \\ &\qquad = \mathbb{E}_x \int_0^{\tau_D(t)} f(X(s))ds \ge \gamma \mathbb{E}_x[\tau_D(t)].\end{aligned}$$

Hence

$$\mathbb{E}_x[\tau_D(t)] \le \frac{1}{\gamma} \sup_{x\in\mathbb{R}^d} f(x) := M. \tag{3.8}$$

Note that M is finite since $f \in C_b^2(\mathbb{R}^d)$. Since $\mathbb{E}_x[\tau_D(t)] \ge t\mathbb{P}_x[\tau_D > t]$, it follows from (3.8) that $t\mathbb{P}_x[\tau_D > t] \le M$. Letting $t \to \infty$, we obtain $\mathbb{P}_x[\tau_D = \infty] = 0$; that is, $\mathbb{P}_x[\tau_D < \infty] = 1$. This yields that $\tau_D(t) \to \tau_D$ a.s. $\mathbb{P}_x$ as $t \to \infty$. Now applying Fatou's lemma, as $t \to \infty$ in (3.8) we obtain

$$\mathbb{E}_x[\tau_D] \le M < \infty. \tag{3.9}$$

This proves the theorem. □

Lemma 3.3. *Let E, D, G be bounded domains in $\mathbb{R}^d$ such that $E \subset \overline{E} \subset D \subset \overline{D} \subset G$, and*

$$u(x) = \mathbb{P}_x(\sigma_E < \tau_G) \quad \text{for} \quad x \in \mathbb{R}^d.$$

Then there exists a positive constant δ such that $u(x) \ge \delta$ for $x \in D$.

Proof. If $\widetilde{\pi}(x,z) = 0$ for some $(x,z) \in \mathbb{R}^d \times \mathbb{R}^d$, then $X(t)$ is a diffusion process. In this case, the conclusion follows immediately from the theory of diffusion processes (see [4, 2]). Therefore, we suppose that $\widetilde{\pi}(x,z) > 0$ for all $(x,z) \in \mathbb{R}^d \times \mathbb{R}^d$. For each $r \in (0, 1/2)$, we define $E_r := \{x \in E : d(x, \partial E) > r\}$. Let $r_0 \in (0, 1/2)$ such that $E_{2r_0} \ne \emptyset$ and $B(x, r_0) \subset G$ for all $x \in D$. Let $y \in D - E$. By Proposition 3.3 in [3],

$$\sum_{s\le t} \mathbf{1}_{\{X(s-)\in G-\overline{E}_{r_0}, X(s)\in E_{2r_0}\}} - \int_0^t \int_{E_{2r_0}} \mathbf{1}_{G-\overline{E}_{r_0}}(X(s))\widetilde{\pi}(X(s), z - X(s))dzds$$

is a $\mathbb{P}_y$-martingale. We deduce that

$$\begin{aligned}
\mathbb{P}_y\Big(X(\tau_{G-\overline{E}_{r_0}}\wedge t)\in E_{2r_0}\Big) &= \mathbb{E}_y\Big[\sum_{s\le\tau_{G-\overline{E}_{r_0}}\wedge t}\mathbf{1}_{\{X(s-)\in G-\overline{E}_{r_0},X(s)\in E_{2r_0}\}}\Big]\\
&= \mathbb{E}_y\Big[\int_0^{\tau_{G-\overline{E}_{r_0}}\wedge t}\int_{E_{2r_0}}\mathbf{1}_{G-\overline{E}_{r_0}}(X(s))\widetilde{\pi}(X(s),z-X(s))dzds\Big]\\
&= \mathbb{E}_y\Big[\int_0^{\tau_{G-\overline{E}_{r_0}}\wedge t}\int_{E_{2r_0}}\widetilde{\pi}(X(s),z-X(s))dzds\Big]\\
&= \mathbb{E}_y\Big[\int_0^{\tau_{G-\overline{E}_{r_0}}\wedge t}\int_{E_{2r_0}}c_1dzds\\
&\ge c_1c_2\mathbb{E}_y[\tau_{G-\overline{E}_{r_0}}\wedge t],
\end{aligned} \tag{3.10}$$

where $c_1=\inf\{\widetilde{\pi}(x,z-x): x\in G-\overline{E}_{r_0}, z\in E_{2r_0}\}>0$ and c_2 is the volume of E_{2r_0} in $\mathbb{R}^d$. Since $\tau_{G-\overline{E}_{r_0}}\ge\tau_{B(y,r_0)}$ and $\mathbb{E}_y[\tau_{B(y,r_0)}]\ge c_3r_0^2$ for some constant $c_3>0$ (see [3, Lemma 3.3]), then $\mathbb{E}_y[\tau_{G-\overline{E}_{2r_0}}]\ge c_3r_0^2$. Hence, for a sufficiently large $t>0$, $\mathbb{E}_y[\tau_{G-\overline{E}_{r_0}}\wedge t]\ge\frac{c_3r_0^2}{2}$. It follows from (3.10) that $\mathbb{P}_y\Big(X(\tau_{G-\overline{E}_{r_0}})\in E_{2r_0}\Big)\ge 0.5c_1c_2c_3r_0^2$; that is, $u(y)\ge 0.5c_1c_2c_3r_0^2$. Since $y\in G-\overline{E}$ is arbitrary, we arrive at $u(y)\ge 0.5c_1c_2c_3r_0^2$ for any $y\in G-E$. By defining $\delta=\min\{1,0.5c_1c_2c_3r_0^2\}$, we obtain $u(x)\ge\delta$ for any $x\in D$. This completes the proof. □

The following criterion for positive recurrence of the process $X(t)$ is given based on the existence of certain Liapunov functions.

Theorem 3.4. *A sufficient condition for the positive recurrence of $X(t)$ with respect to $D\subset\mathbb{R}^d$ is: The operator $\mathcal{L}$ satisfies the following condition* (L) *with respect to D.*

(L) *There exists a nonnegative function $V\in C^2(\mathbb{R}^d)$ satisfying*

$$\mathcal{L}V(x)\le -1 \text{ for any } x\in D^c. \tag{3.11}$$

Proof. Assume that $V\in C^2(\mathbb{R}^d)$ and $\mathcal{L}V(x)\le -1$ for any $x\in D^c$. Take a sufficiently large positive integer n_0 so that $D\subset B(0,n_0)$. Fix a point $x\in D^c$. For any $t>0$ and $n\ge n_0$, we define

$$\sigma_D^{(n)}(t)=\min\{\sigma_D,t,\beta_n\},$$

where β_n is the first exit time from $B(0,n)$ and σ_D is the first entrance time to D. Let $f_n:\mathbb{R}^d\mapsto\mathbb{R}$ be a smooth cut-off function that takes values in $[0,1]$ satisfying $f_n=1$ on $B(0,n)$ and $f_n=0$ outside of $B(0,n+1)$. Then $V_n:=f_nV\in C_b^2(\mathbb{R}^d)$. Moreover, $0\le V_n(x)\le V(x)$ for $x\in\mathbb{R}^d$. It follows from (3.11) that

$$\mathcal{L}V_n(y)\le\mathcal{L}V(y)\le -1\quad\text{for}\quad y\in B(0,n)-D.$$

Dynkin's formula implies that

$$\begin{aligned}\mathbb{E}_x V_n\left(X\left(\sigma_D^{(n)}(t)\right)\right) - V_n(x) \\ = \mathbb{E}_x \int_0^{\sigma_D^{(n)}(t)} \mathcal{L}V_n\left(X(s)\right) ds \leq -\mathbb{E}_x\left[\sigma_D^{(n)}(t)\right].\end{aligned}$$

Note that the function V_n is nonnegative; hence we have $\mathbb{E}_x\left[\sigma_D^{(n)}(t)\right] \leq V_n(x) = V(x)$. Meanwhile, since $\beta_n \to \infty$ a.s as $n \to \infty$, $\sigma_D^{(n)}(t) \to \sigma_D(t)$ a.s as $n \to \infty$, where $\sigma_D(t) = \min\{\sigma_D, t\}$. By virtue of Fatou's lemma, we obtain $\mathbb{E}_x[\sigma_D(t)] \leq V(x)$. Now the argument after (3.8) in the proof of Lemma 3.2 yields that

$$\mathbb{E}_x[\sigma_D] \leq V(x) < \infty. \tag{3.12}$$

Since $x \in D^c$ is arbitrary, then $X(t)$ is positive recurrent with respect to D. □

Theorem 3.5. *Let $D \subset \mathbb{R}^d$ be a bounded domain and suppose that $\mathcal{L}$ satisfies condition* (L) *with respect to D. Then for any bounded domain $E \subset \mathbb{R}^d$, $X(t)$ is positive recurrent with repsect to E; that is, $\mathbb{E}_x[\sigma_E] < \infty$ for any $x \in E^c$.*

Proof. Since $\mathcal{L}$ satisfies condition (L) with respect to D,

$$\mathbb{E}_y[\sigma_D] < \infty \quad \text{for any} \quad y \in D^c. \tag{3.13}$$

Without loss of generality, we suppose that $\overline{E} \subset D$. Let G be a domain in $\mathbb{R}^d$ such that $\overline{D} \subset G$. Define a sequence of stopping times by $\tau_0 = 0$, $\tau_1 = \inf\{t > 0 : X(t) \in G^c\}$ and for $n = 1, 2, \dots,$

$$\tau_{2n} = \inf\{t > \tau_{2n-1} : X(t) \in D\}, \quad \tau_{2n+1} = \inf\{t > \tau_{2n} : X(t) \in G^c\}. \tag{3.14}$$

It follows from (3.13), Lemma 3.2, and the strong Markov property that for each $x \in D$ and $n = 1, 2, \dots$, $\tau_n < \infty$ $\mathbb{P}_x$ a.s.. Define

$$A_n = \{X(t) \in E \text{ for some } t \in [\tau_{2n}, \tau_{2n+1})\}, \quad n = 0, 1, 2, \dots. \tag{3.15}$$

By Lemma 3.3, there exists $\delta > 0$ such that $\inf_{x \in D} \mathbb{P}_x(\sigma_E < \tau_G) > \delta$. It follows that $\mathbb{P}_x(A_0^c) \leq 1 - \delta$. By the strong Markov property, induction on n yields

$$\mathbb{P}_x\left(\bigcap_{k=0}^{n-1} A_k^c\right) \leq (1-\delta)^n, \quad n = 1, 2, \dots, x \in D. \tag{3.16}$$

To prove the theorem, it suffices to prove (3.13) for $x \in \big(D - \overline{E}\big)$. To this end, we claim that

$$M_2 := \sup_{y \in D} \mathbb{E}_y[\tau_2] < \infty. \tag{3.17}$$

By Lemma 3.2, we have $M_1 := \sup_{y \in D} \mathbb{E}_y[\tau_1] < \infty$. Meanwhile, by (3.12) in the proof of Theorem 3.4, we obtain $\mathbb{E}_y[\sigma_D] \leq V(y) \quad \text{for} \quad y \in G^c$. Hence to prove (3.17), it suffices to show that

$$\sup_{y \in D} \int_{G^c} V(z) \mathbb{P}_{\tau_1}(y, dz) < \infty,$$

where $\mathbb{P}_{\tau_1}(y, \cdot)$ is the distribution of $X^y(\tau_1)$. Since V is bounded on compact sets, it is enough if we can find a open ball $B(0, R)$ with R sufficiently large such that $\{y : d(y, G) < 2\} \subset B(0, R)$ and

$$\sup_{y \in D} \int_{B(0,R)^c} V(z) \mathbb{P}_{\tau_1}(y, dz) < \infty. \tag{3.18}$$

Let a point $x^* \in \partial G$. Then for any $y \in G$ and $z \in B(0, R)^c$, there is a sequence $\{x_i : i = 0, ..., \widetilde{n}\}$ such that $x_0 = y, x_{\widetilde{n}} = x^*$, $|x_i - x_{i-1}| < 1/2$ and $x_i \in \overline{G}$ for $i = 1, \dots, \widetilde{n}$. Since G is bounded, $\widetilde{n}$ can be independent of y. By assumption (A4), we have

$$\widetilde{\pi}(x_{i-1}, z - x_{i-1}) \leq \alpha_{1/2} \widetilde{\pi}(x_i, z - x_i), \quad i = 1, \dots, \widetilde{n}.$$

Thus, there is a positive constant $K = \alpha_{1/2}^{\widetilde{n}}$, depending only on G such that

$$\widetilde{\pi}(x, z - x) \leq K \widetilde{\pi}(x^*, z - x^*) \quad \text{for} \quad y \in G, z \in B(0, R)^c.$$

Let $y \in D$ and $A \subset B(0, R)^c$. By Proposition 3.3 in [3],

$$\sum_{s \leq t} \mathbf{1}_{\{X(s-) \in G, X(s) \in A\}} - \int_0^t \int_A \mathbf{1}_G(X(s)) \widetilde{\pi}(X(s), z - X(s)) dz ds,$$

is a $\mathbb{P}_y$-martingale. We deduce that

$$\begin{aligned} \mathbb{P}_y\left(X(\tau_1 \wedge t) \in A\right) &= \mathbb{E}_y\Big[\sum_{s \leq \tau_1 \wedge t} \mathbf{1}_{\{X(s-) \in G, X(s) \in A\}}\Big] \\ &= \mathbb{E}_y\Big[\int_0^{\tau_1 \wedge t} \mathbf{1}_{\{X(s) \in G\}} \int_A \widetilde{\pi}(X(s), z - X(s)) dz ds\Big] \\ &\leq K \mathbb{E}_y\Big[\int_0^{\tau_1 \wedge t} \int_A \widetilde{\pi}(x^*, z - x^*) dz ds\Big] \\ &\leq K \mathbb{E}_y\left(\tau_1 \wedge t\right) \mu(A), \end{aligned} \tag{3.19}$$

where μ is a measure on $B(0,R)^c$ with density $\widetilde{\pi}(x^*, z-x^*)$. Using assumption (A3) and the fact that $A \cap B(0,R) \subset B(0,R)^c \cap B(0,R) = \emptyset$, we have $\mu(A) < \infty$. Letting $t \to \infty$ and using Fatou's lemma on the left-hand side and the dominated convergence theorem on the right-hand side in (3.19), we have

$$\mathbb{P}_y\left(X(\tau_1) \in A\right) \leq K\mathbb{E}_y[\tau_1]\mu(A) \leq KM_1\mu(A),$$

where $M_1 = \sup_{z\in D} \mathbb{E}_z[\tau_1]$. Hence $\mathbb{P}_{\tau_1}(y, A) \leq KM_1\mu(A)$. It follows that

$$\begin{aligned}\int_{B(0,R)^c} V(z)\mathbb{P}_{\tau_1}(y, dz) &\leq KM_1 \int_{B(0,R)^c} V(z)\widetilde{\pi}(x^*, z-x^*)dz \\ &= KM_1 \int_{B(0,R)^c - x^*} V(z+x^*)\widetilde{\pi}(x^*, z)dz.\end{aligned} \tag{3.20}$$

Take R sufficiently large such that $B(0,R)^c - x^* \subset \{z \in \mathbb{R}^d : |z| > 1\}$. Since $\mathcal{L}$ satisfies condition (L) with respect to D and $x^* \notin D$, we have $\mathcal{L}V(x^*) < \infty$. This leads to the finiteness of the last term in (3.20). The desired inequality (3.18) then follows. Therefore we have

$$\begin{aligned}\mathbb{E}_x[\sigma_E] &= \sum_{n=0}^{\infty} \mathbb{E}_x[\sigma_E]\mathbf{1}_{[\tau_{2n} \leq \sigma_E < \tau_{2n+2}]} \\ &\leq \sum_{n=0}^{\infty} \mathbb{P}_x[\tau_{2n} \leq \sigma_E < \tau_{2n+2}]\mathbb{E}_x[\tau_{2n+2}] \\ &\leq \sum_{n=0}^{\infty} \mathbb{P}_x[\tau_{2n} \leq \sigma_E < \tau_{2n+1}] \sum_{k=0}^{n} \mathbb{E}_x\left(\tau_{2k+2} - \tau_{2k}\right) \\ &\leq \sum_{n=0}^{\infty} (1-\delta)^n (n+1)M_2 < \infty.\end{aligned}$$

This completes the proof of theorem. □

4 FURTHER REMARKS

This paper focused on positive recurrence of a class of jump diffusion processes. A criterion for positive recurrence was derived. The results obtained here will help future studies on controlled jump diffusion systems. A problem of interest is to develop practical criteria for positive recurrence of diffusions with jumps.

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